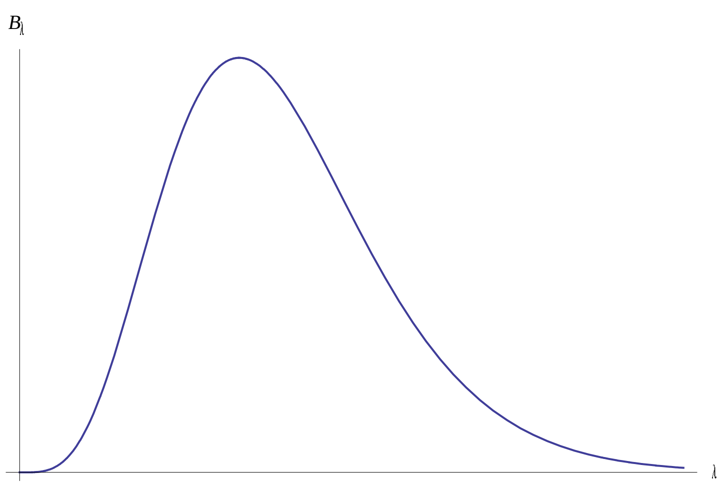
Teacher notes
Topic B

Planck's black body radiation law – last post in 2023.

A black body is a theoretical body that absorbs all the radiation incident on it reflecting none, a perfect absorber. To be in thermal equilibrium (see the end) means that as much intensity is radiated as is absorbed and so a black body is also a perfect emitter, emitting the maximum possible intensity at a given temperature. The radiation emitted is electromagnetic radiation, i.e. photons with a range of wavelengths from zero to infinity. But not the same intensity is emitted at different wavelengths. Planck experimentally discovered that the intensity radiated per unit wavelength is given by

$$B_\lambda = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}$$

B_λ is called spectral intensity or spectral radiance. The intensity (in W m^{-2}) radiated in a wavelength interval dλ is $B_\lambda d\lambda$. Hence the total intensity radiated by the body is $I = \int_0^\infty B_\lambda \, d\lambda$ and is given by the area under the graph giving the variation of B_λ with λ:



To derive this law, Planck had to assume that the energy of the photons was a constant times the frequency. The constant is what we now call the Planck constant *h*.

It must be stressed that it is not correct to label the vertical axis of this graph as "intensity" or even "relative intensity". As we will see below, intensity is the total radiated power per unit area and is given by $I=\sigma T^4$ which shows no dependence on wavelength. Thus, a graph of intensity versus wavelength would be a horizontal straight line.

We can calculate the radiated intensity as follows:

Calling $x=\dfrac{hc}{\lambda kT}$ we get

$$B_\lambda = \frac{2\pi(kT)^5}{h^4c^3}\frac{x^5}{e^x-1}$$

The total intensity radiated is

$$I=\int_0^\infty B_\lambda\, d\lambda = \int_0^\infty \frac{2\pi(kT)^5}{h^4c^3}\frac{x^5}{e^x-1}\frac{d\lambda}{dx}dx$$

$$I=\int_\infty^0 \frac{2\pi(kT)^5}{h^4c^3}\frac{x^5}{e^x-1}\frac{hc}{kT}\left(-\frac{1}{x^2}\right)dx$$

$$I=\frac{2\pi(kT)^4}{h^3c^2}\int_0^\infty \frac{x^3}{e^x-1}dx$$ ($\int_0^\infty \frac{x^3}{e^x-1}dx=\frac{\pi^4}{15}$ is a standard integral-see below)

$$I=\frac{2\pi(kT)^4}{h^3c^2}\frac{\pi^4}{15}$$

$$I=\frac{2\pi^5k^4}{15h^3c^2}T^4$$

$$I=\sigma T^4$$

where the Stefan-Boltzmann constant is defined to be

$$\sigma=\frac{2\pi^5k^4}{15h^3c^2}=\frac{2\pi^5\times(1.38\times10^{-23})^4}{15\times(6.63\times10^{-34})^3\times(2.998\times10^8)^2}=5.67\times10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4}.$$

The radiated intensity $I=\sigma T^4$ is the area under the black body curve.

From the Planck law follows Wien's law which relates temperature to the wavelength at which the black body curve has its maximum: $\lambda_0 T=\text{constant}=2.9\times10^{-3}\ \text{K m}$. This is done as follows: we differentiate $B_\lambda=\dfrac{2\pi hc^2}{\lambda^5}\dfrac{1}{e^{\frac{hc}{\lambda kT}}-1}$ with respect to the wavelength to get

$$\frac{d}{d\lambda}B_\lambda = \frac{d}{d\lambda}\frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}$$
$$=2\pi hc^2\left(\frac{-5}{\lambda^6}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}-\frac{1}{\lambda^5}\frac{1}{(e^{\frac{hc}{\lambda kT}}-1)^2}e^{\frac{hc}{\lambda kT}}\frac{hc}{kT}(-\frac{1}{\lambda^2})\right)$$
$$=\frac{2\pi hc^2}{e^{\frac{hc}{\lambda kT}}-1}\frac{1}{\lambda^6}\left(-5+\frac{1}{(e^{\frac{hc}{\lambda kT}}-1)}e^{\frac{hc}{\lambda kT}}\frac{hc}{\lambda kT}\right)$$
$$=0$$

Hence,

$$-5+\frac{1}{(e^{\frac{hc}{\lambda kT}}-1)}e^{\frac{hc}{\lambda kT}}\frac{hc}{\lambda kT}=0$$

Call $x=\frac{hc}{\lambda kT}$, then

$$-5+\frac{1}{(e^x-1)}e^x x=0$$
$$xe^x-5(e^x-1)=0$$

The numerical solution of this equation is x = 4.96511. Hence

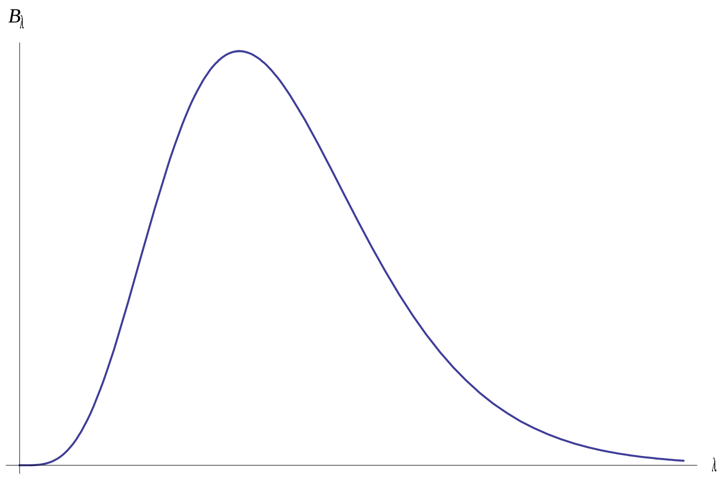
$x=\frac{hc}{\lambda kT}=4.96511$ leading to

$$\lambda T=\frac{hc}{k\times 4.96511}=2.9\times 10^{-3}\quad \text{K m}$$

It is crucial that black body radiation means that the emitted photons are in thermal equilibrium with the surroundings. This means that a perfect black body is an idealization. Thermal equilibrium with the surroundings would mean no difference in temperature (no temperature gradient) and so the net intensity radiated would be zero. As much radiation is emitted as is absorbed. The radiation emitted by a star, like the Sun, comes from the top layer of the Sun, the photosphere which is essentially uniform in temperature (zero temperature gradient). In addition, the Sun is opaque so that photons incident on the Sun are absorbed making the Sun an almost perfect absorber. For these reasons, stars are a very good approximation to an ideal black body, unlike, for example, a gas flame which does not absorb photons incident on it. Precisely because stars are only approximate black bodies their spectra show deviations from ideal black body behavior. In the Sun for example, we have absorption lines and at various wavelengths the intensity emitted is sometimes above and sometimes below the ideal black body curve at T = 5780 K. This is because the radiation emitted also includes radiation from different layers of the Sun which have temperatures different from 5780 K.

Questions

The black body curve for a body of area A and temperature T is given by the following graph.



(a) What happens to the graph if A is doubled?
(b) The temperature T of the body increases. How does the graph change?
(c) Which is hotter, a red star or a blue star?
(d) The Sun has a peak in its black body spectrum at a wavelength of about 500 nm. What is the surface temperature of the Sun?
(e) The wavelength in (d) is a blue-green wavelength. Is the color of the Sun blue-green then? Investigate this a bit on your own.

Derivation of the integral

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \int_0^\infty \frac{x^3 e^{-x}}{1 - e^{-x}}\,dx$$

$$\frac{1}{1 - e^{-x}} = \sum_{n=0}^\infty e^{-nx}$$

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \int_0^\infty \sum_{n=0}^\infty x^3 e^{-x} e^{-nx}\,dx = \int_0^\infty \sum_{n=0}^\infty x^3 e^{-(n+1)x}\,dx$$

$$= \sum_{n=0}^\infty \int_0^\infty x^3 e^{-(n+1)x}\,dx$$

Change variables to $t = (n+1)x \Rightarrow dt = (n+1)dx$ so that

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx = \sum_{n=0}^{\infty}\int_0^\infty \frac{t^3e^{-t}}{(n+1)^4}\,dt$$

$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)^4}\int_0^\infty t^3e^{-t}\,dt$$

$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)^4}\Gamma(4) = \sum_{n=1}^{\infty}\frac{1}{n^4}\Gamma(4)$$

$$= \zeta(4)\Gamma(4) \quad \text{where } \varsigma(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \quad \text{is the Riemann zeta function}$$

$$= \frac{\pi^4}{90}\times 3!$$

$$= \frac{\pi^4}{15}$$

Answers to questions

(a) No change.

(b) The new curve is above the original curve and the peak is shifted to the left.

(c) Blue has shorter wavelength and by Wien's law a higher temperature.

(d) From $\lambda_0 T = \text{constant} = 2.9\times 10^{-3}$ K m, we get $T = \dfrac{2.9\times 10^{-3}}{5.0\times 10^{-7}} = 5800$ K.